

Vertex Lie algebras, vertex Poisson algebras and vertex algebras

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ABSTRACT. The notions of vertex Lie algebra and vertex Poisson algebra are presented and connections among vertex Lie algebras, vertex Poisson algebras and vertex algebras are discussed.

1. Introduction

Vertex (operator) algebras (see [B], [FLM]) which are a family of new “algebras” are known essentially to be chiral algebras in two-dimensional quantum field theory. The new algebras are closely related to classical Lie algebras of certain types. On one hand, from [B] one can obtain a Lie algebra V/DV from any vertex algebra V where D is a canonical endomorphism of V . (The famous Monster Lie algebra is a subalgebra of V/DV for a suitable V .) On the other hand, one can construct vertex (operator) algebras from certain highest weight representations of familiar infinite dimensional Lie algebras such as the Virasoro algebra, affine Kac-Moody algebras, Heisenberg algebras (cf. [DL], [FF1], [FZ], [Li2], [MP]).

In this paper we define a notion of what we call vertex Lie algebra to unify the familiar infinite dimensional Lie algebras. The definition of a vertex Lie algebra is motivated by the Virasoro algebra, affine algebras and the notion of vertex algebra. Roughly speaking, the notion of vertex Lie algebra is a “stringy” analogue of the notion of Lie algebra, which generalizes the Virasoro algebra and affine Lie algebras. A vertex Lie algebra defined in this paper is a Lie algebra whose underlying vector space is essentially the loop space of a certain vector space U and it is a kind of “affine algebra” based on vector space U instead of a finite dimensional Lie algebra. So for every vector u in the base vector space, one can form the field or vertex operator $u(z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1}$. In the case of the Virasoro algebra, U is either one dimensional or two dimensional depending on the center being zero or not. (The Virasoro algebra is not a classical affine algebra in any sense.) We also

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define a notion of vertex Poisson algebra. We study the connection among vertex Lie algebras, vertex Poisson algebras and vertex algebras.

The notion of vertex Lie algebras not only unifies the Virasoro algebra, affine Lie algebras, loop algebras and other important infinite dimensional Lie algebras, but also provides new examples of vertex algebras via their “highest weight representations.” We prove that for each complex number there is a “highest weight module” (based a polar decomposition) for the vertex Lie algebra which has the structure of a vertex algebra. We also show how a restricted module for the vertex Lie algebra has a natural module structure for certain vertex algebras constructed from the vertex Lie algebra. So it is very important to construct new vertex Lie algebras.

Vertex Poisson Lie algebras are a special class of vertex Lie algebras whose base vector spaces are Poisson algebras satisfying additional axioms. Closely related to vertex Poisson algebras are Poisson brackets studied in [DFN], [DN] and [GD]. We were amazed to find that local Poisson brackets introduced in [DFN] was so close to Borchers’ commutator formula in the theory of vertex algebras. Our first exercise is to make local Poisson brackets precise in terms of formal variables where delta functions are formal series instead of distributions. Differential geometric Poisson brackets introduced in [DN], where interesting connections between differential geometric objects (connection, curvature and symplectic structure) and algebraic objects (Lie algebra, commutative associative algebra) have been established, provide a lot of examples of vertex Poisson Lie algebras. At the end of Section 3, we quote several interesting results from [DFN], [DN] and [Po] regarding to differential geometric Poisson brackets.

In [BD], a notion of coisson algebra was defined in terms of algebraic geometry where as designated by the authors, coisson is the combination of the two words chiral and poisson. Later, a notion of vertex Poisson algebra was defined in [EF] in terms of associative rings. (The relation between coisson algebras and vertex Poisson algebras was discussed in [EF].) The notion of vertex Poisson algebra presented here is defined in terms of formal calculus and classical (Lie and associative) algebras. Presumably, our notion is essentially the same as that of [EF].

Most of this work was carried out in the late 96 and early 97. During this time, Kac’s book [K2] appeared where similar results to ours had been obtained. In particular, a notion of conformal algebra was introduced. A conformal algebra satisfies a set of axioms which are certain modifications of those for a vertex algebra and a conformal algebra naturally gives to a vertex Lie algebra. A notion of vertex Lie algebra was also introduced in [Pr]. It seems that the notion of vertex Lie algebra in [Pr] the notion of conformal algebra [K2] are the same. There are certain overlaps between this work and [Pr]. In particular, Theorem 4.8 and Lemma 5.3 were also obtained in [Pr].

The paper is organized as follows. Section 2 is about formal calculus. In Section 3 we formulate the notions of local vertex Lie algebra and local vertex Poisson algebra and present some results. We also give several examples including the Witt algebra, Virasoro algebra, loop algebras, affine algebras to illustrate the concepts. In particular, we discuss vertex Poisson algebras for the base vector spaces being the Poisson algebras $S(\mathfrak{g})$ which is the symmetric algebra of Lie algebra \mathfrak{g} . In Section 4 we construct vertex algebras and their modules from vertex Lie algebras by using the frame work of local system introduced in [Li2]. Since it has been an important problem in the theory of vertex algebras to construct new examples of

vertex algebras, the vertex algebra construction based on vertex Lie algebras is of certain importance. Section 5 is devoted to the study of Lie algebras and Poisson algebras associated to vertex algebras. Using the results obtained in Section 4 we construct a vertex algebra $V^{[l]}$ for any vertex algebra V and any complex number l . In particular, if V is a vertex operator algebra of CFT type we construct a new vertex operator algebra $V^{[l]}$ whose central charge is lc where c is the central charge of V . We also investigate the Poisson algebra $P_2(V)$ (see [Zh]) for vertex algebra constructed from a vertex Lie algebra. We show that for any Lie algebra \mathfrak{g} there exists a vertex algebra V such that $P_2(V)$ is isomorphic to $S(\mathfrak{g})$ as Poisson algebras.

2. Calculus of formal variables

In this section we review some formal variable notations and the fundamental properties of delta function from [FLM] and we also formulate some simple results which will be used later.

Throughout the whole paper, $x, x_0, x_1, \dots, y, y_0, y_1, \dots, z, z_0, z_1, \dots$ are independent commuting formal variables. The symbols $\mathbb{Z}, \mathbb{N}, \mathbb{C}$ stand for the integers, the nonnegative integers and the complex numbers. Vector spaces are over the ground field \mathbb{C} .

For any vector space U , following [FLM] we set

$$(2.1) \quad U[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \mid a_n \in U \right\},$$

$$(2.2) \quad U((x)) = \left\{ \sum_{n \geq m} a_n x^n \mid m \in \mathbb{Z}, a_n \in U \right\},$$

$$(2.3) \quad U[[x]] = \left\{ \sum_{n=0}^{\infty} a_n x^n \mid a_n \in U \right\},$$

$$(2.4) \quad U[x, x^{-1}] = \left\{ \sum_{n=m}^k a_n x^n \mid m, k \in \mathbb{Z}, a_n \in U \right\}.$$

The formal delta-function is defined to be the formal series $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$ (an element of $\mathbb{C}[[x, x^{-1}]]$) and its fundamental property is

$$(2.5) \quad f(x)\delta(x) = f(1)\delta(x) \quad \text{for } f(x) \in U[x, x^{-1}].$$

Furthermore

$$(2.6) \quad f(x)\delta\left(\frac{x}{y}\right) = f(y)\delta\left(\frac{x}{y}\right) \quad \text{for } f(x) \in U[[x, x^{-1}]].$$

For any $n \in \mathbb{Z}$, $(x+y)^n$ is defined to be the formal series $\sum_{i=0}^{\infty} \binom{n}{i} x^{n-i} y^i$, where $\binom{n}{i} = \frac{1}{i!} n(n-1) \cdots (n+1-i)$. We also have the following identities in three formal variables (see [FLM]):

$$(2.7) \quad x^{-1}\delta\left(\frac{y-z}{x}\right) = y^{-1}\delta\left(\frac{x+z}{y}\right),$$

$$(2.8) \quad x^{-1}\delta\left(\frac{y-z}{x}\right) - x^{-1}\delta\left(\frac{z-y}{-x}\right) = z^{-1}\delta\left(\frac{y-x}{z}\right).$$

By using the well-known Taylor formula and (2.7) we get

$$e^{-z\partial/\partial y}x^{-1}\delta\left(\frac{y}{x}\right) = x^{-1}\delta\left(\frac{y-z}{x}\right) = y^{-1}\delta\left(\frac{x+z}{y}\right) = e^{z\partial/\partial x}y^{-1}\delta\left(\frac{x}{y}\right).$$

This amounts to that

$$(2.9) \quad \left(-\frac{\partial}{\partial x}\right)^k y^{-1}\delta\left(\frac{x}{y}\right) = \left(\frac{\partial}{\partial y}\right)^k y^{-1}\delta\left(\frac{x}{y}\right) \quad \text{for } k \in \mathbb{N}.$$

Later, we shall frequently use the derivatives of $y^{-1}\delta(\frac{x}{y})$. For convenience, we set

$$(2.10) \quad \Delta^{(k)}(x, y) = \left(\frac{\partial}{\partial x}\right)^k y^{-1}\delta\left(\frac{x}{y}\right) = \left(-\frac{\partial}{\partial y}\right)^k y^{-1}\delta\left(\frac{x}{y}\right)$$

for $k \in \mathbb{N}$. If $k = 0$ we simply write $\Delta(x, y)$ for $\Delta^{(0)}(x, y) = y^{-1}\delta(\frac{x}{y})$.

For $f(x) \in \mathbb{C}[[x, x^{-1}]]$, since $f(x)\Delta(x, y) = f(y)\Delta(x, y)$, we have

$$(2.11) \quad \begin{aligned} f(x)\Delta^{(k)}(x, y) &= \left(-\frac{\partial}{\partial y}\right)^k f(x)\Delta(x, y) \\ &= (-1)^k \left(\frac{\partial}{\partial y}\right)^k (f(y)\Delta(x, y)) \\ &= (-1)^k \sum_{j=0}^k \binom{k}{j} f^{(k-j)}(y)\Delta^{(j)}(x, y). \end{aligned}$$

LEMMA 2.1. *Let $m, n \in \mathbb{N}$. If $m \leq n$ we have*

$$(2.12) \quad (x - y)^m \Delta^{(n)}(x, y) = \binom{-n}{m} m! \Delta^{(n-m)}(x, y).$$

If $m > n$ we have

$$(2.13) \quad (x - y)^m \Delta^{(n)}(x, y) = 0.$$

PROOF. Since $(x - y)\Delta(x, y) = 0$, we have

$$0 = \left(\frac{\partial}{\partial x}\right)^n ((x - y)\Delta(x, y)) = (x - y)\Delta^{(n)}(x, y) + n\Delta^{(n-1)}(x, y).$$

Thus $(x - y)\Delta^{(n)}(x, y) = -n\Delta^{(n-1)}(x, y)$ for $n \geq 1$. Then (2.12) follows from induction on m immediately. Furthermore, (2.13) follows from (2.12) and the fact that $(x - y)\Delta^{(0)}(x, y) = 0$. \square

Recall the following result from [Li2] (cf. [FLM], Proposition 8.1.3):

LEMMA 2.2. *Let U be any vector space and let $f_i(y) \in U[[y, y^{-1}]]$ for $i = 0, 1, \dots, n$. Then*

$$(2.14) \quad f_0(y)\Delta(x, y) + f_1(y)\Delta^{(1)}(x, y) + \dots + f_n(y)\Delta^{(n)}(x, y) = 0$$

if and only if $f_i(y) = 0$ for all i . Consequently, the expression of an element $h(x, y)$ of $U[[x, y, x^{-1}, y^{-1}]]$ as a finite sum $\sum_{i=0}^n g_i(y)\Delta^{(i)}(x, y)$ is unique if it exists.

REMARK 2.3. Similarly, if U is a vector space and $f_i(x) \in U[[x, x^{-1}]]$, $i = 0, 1, \dots, n$, then

$$(2.15) \quad f_0(x)\Delta(x, y) + f_1(x)\Delta^{(1)}(x, y) + \dots + f_n(x)\Delta^{(n)}(x, y) = 0$$

if and only if $f_i(x) = 0$ for all i . Consequently, the expression of an element $h(x, y)$ of $U[[x, y, x^{-1}, y^{-1}]]$ as a finite sum $\sum_{i=0}^n g_i(x)\Delta^{(i)}(x, y)$ is unique if it exists.

The following result can also be found in [K2] (cf. [FLM], Proposition 8.1.3): (Our proof is slightly different from that of [K2].)

PROPOSITION 2.4. *Let U be any vector space and let $f(x, y) \in U[[x, y, x^{-1}, y^{-1}]]$, a formal series in x, y with coefficients in U . Then $(x - y)^{k+1}f(x, y) = 0$ for some nonnegative integer k if and only if $f(x, y) = \sum_{i=0}^k f_i(y)\Delta^{(i)}(x, y)$ for some $f_0(y), \dots, f_k(y) \in U[[y, y^{-1}]]$.*

PROOF. Since $(x - y)^m\Delta^{(n)}(x, y) = 0$ for any nonnegative integers $m > n$, one direction is clear. We shall prove the other direction by using induction on k . Suppose that $(x - y)f(x, y) = 0$. Let $f(x, y) = \sum_{m, n \in \mathbb{Z}} a(m, n)x^m y^n$. Then we have $a(m + 1, n) = a(m, n + 1)$ for any $m, n \in \mathbb{Z}$, so that $a(m, n) = a(m + n, 0)$ for $m, n \in \mathbb{Z}$. Then $f(x, y) = f_0(y)\delta\left(\frac{x}{y}\right)$, where $f_0(y) = \sum_{n \in \mathbb{Z}} a(n, 0)y^n$. Thus it is true for $k = 0$.

Now, suppose that it is true for k and that a formal series $f(x, y)$ satisfies the relation $(x - y)^{k+2}f(x, y) = 0$. Set $g(x, y) = (x - y)f(x, y)$. Then $(x - y)^{k+1}g(x, y) = 0$. By the inductive hypothesis, there are $g_0(y), \dots, g_k(y) \in U[[y, y^{-1}]]$ such that

$$g(x, y) = \sum_{j=0}^k g_j(y)\Delta^{(j)}(x, y).$$

Set

$$F(x, y) = \sum_{j=0}^k \frac{1}{j+1} g_j(y)\Delta^{(j+1)}(x, y).$$

Since $(x - y)\Delta^{(n)}(x, y) = -n\Delta^{(n-1)}(x, y)$ for any positive integer n , we have

$$\begin{aligned} & (x - y)(f(x, y) + F(x, y)) \\ &= \sum_{j=0}^k \left(g_j(y)\Delta^{(j)}(x, y) + \frac{1}{j+1}(x - y)g_j(y)\Delta^{(j+1)}(x, y) \right) = 0. \end{aligned}$$

From the base step, we have $f(x, y) + F(x, y) = g(y)\Delta(x, y)$ for some $g(y) \in U[[y, y^{-1}]]$. Then the inductive step follows immediately. This concludes the proof. \square

REMARK 2.5. Set

$$(2.16) \quad A(x, y) = \sum_{n \in \mathbb{N}} \mathbb{C}[[x, x^{-1}]]\Delta^{(n)}(x, y) \subset \mathbb{C}[[x, y, x^{-1}, y^{-1}]].$$

From (2.11) we also have

$$(2.17) \quad A(x, y) = \sum_{n \in \mathbb{N}} \mathbb{C}[[y, y^{-1}]]\Delta^{(n)}(x, y).$$

In view of Proposition 2.4, $A(x, y)$ can be canonically defined by

$$A(x, y) = \{f(x, y) \in \mathbb{C}[[x, y, x^{-1}, y^{-1}]] \mid (x - y)^k f(x, y) = 0 \text{ for some } k \in \mathbb{N}\}.$$

It follows from Lemma 2.2 (with $U = \mathbb{C}$) that

$$(2.18) \quad A(x, y) = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[[x, x^{-1}]] \Delta^{(n)}(x, y) = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[[y, y^{-1}]] \Delta^{(n)}(x, y).$$

We shall need the following result (cf. [MN], Lemma 1.1.1):

LEMMA 2.6. *Let U be a vector space and let*

$$f(x, y) \in \mathbb{C}((x)) \otimes U[[y, y^{-1}]], \text{ or } \mathbb{C}((y)) \otimes U[[x, x^{-1}]].$$

Then $(x - y)^k f(x, y) = 0$ for some nonnegative integer k if and only if $f(x, y) = 0$.

PROOF. Suppose that $f(x, y) \in \mathbb{C}((x)) \otimes U[[y, y^{-1}]]$. Then, for any $n \in \mathbb{Z}$,

$$(-y + x)^n f(x, y) \text{ exists}$$

in $U[[x, x^{-1}, y, y^{-1}]]$. Then from [FLM] (Chapter 2) we have

$$(2.19) \quad (-y + x)^m ((-y + x)^n f(x, y)) = (-y + x)^{m+n} f(x, y)$$

for any $m, n \in \mathbb{Z}$. It follows immediately that $(x - y)^k f(x, y) = 0$ for some nonnegative integer k if and only if $f(x, y) = 0$. Similarly, this is true for $f(x, y) \in \mathbb{C}((y)) \otimes U[[x, x^{-1}]]$. \square

REMARK 2.7. Let L be any Lie algebra and let $\psi(x), \phi(x) \in L[[x, x^{-1}]]$. Then by Proposition 2.4, there exists a nonnegative integer k such that

$$(2.20) \quad (x - y)^k \psi(x) \phi(y) = (x - y)^k \phi(y) \psi(x)$$

if and only if there exist $f_0(y), \dots, f_k(y) \in L[[y, y^{-1}]]$ such that

$$(2.21) \quad [\psi(x), \phi(y)] = \sum_{i=0}^k f_i(y) \Delta^{(i)}(x, y).$$

Furthermore, it follows from Lemma 2.2 that such $f_0(y), \dots, f_k(y)$ are uniquely determined by $\psi(x)$ and $\phi(x)$. Suppose $\psi(x) \in L[x, x^{-1}]$. Then it follows from Lemma 2.6 that (2.20) holds if and only if $\psi(x) \phi(y) = \phi(y) \psi(x)$.

3. Local vertex Lie algebra and vertex Poisson algebras

In this section we formulate certain notions of local vertex Lie algebra and local vertex Poisson algebra, and give several consequences of the definition. A notion of a vertex Lie algebra was also independently defined in [Pr] and [EF], and a notion of a vertex Poisson algebra has been defined in [EF] (and in [BD]). The notion of a conformal algebra defined in [K2] is also closely related to the notion of a vertex Lie algebra.

DEFINITION 3.1. (1) A *local vertex Lie algebra* is a quadruple (L, U, d, ρ) consisting of a Lie algebra L , a vector space U and a linear map ρ from $L(U) = U \otimes \mathbb{C}[t, t^{-1}]$ by definition onto L such that $\ker \rho = \text{im } \hat{d}$, where $\hat{d} = d \otimes 1 + 1 \otimes \frac{d}{dt}$ and d is a partially defined linear map from U to U , and that for any $u, v \in U$, there exist finitely many $f_0(u, v), \dots, f_r(u, v) \in U$ and nonnegative integers $k_0, \dots, k_r, l_0, \dots, l_r$ (where r depends on both u and v) such that

$$(3.1) \quad [u(x), v(y)] = f_0(u, v)^{(k_0)}(y) \Delta^{(l_0)}(x, y) + \dots + f_r(u, v)^{(k_r)}(y) \Delta^{(l_r)}(x, y),$$

where for $i \in \mathbb{N}$, $u \in U$,

$$u^{(i)}(x) = \left(\frac{d}{dx}\right)^i \sum_{n \in \mathbb{Z}} u(n)x^{-n-1} = \sum_{n \in \mathbb{Z}} \binom{-n-1}{i} i! u(n)x^{-n-i-1}$$

and $u(n) = \rho(u \otimes t^n)$. We simply write $L = (L, U, d, \rho)$.

(2) A \mathbb{Z} -graded vertex Lie algebra is a vertex Lie algebra L over U such that L is a \mathbb{Z} -graded Lie algebra and $U = \bigoplus_{n \in \mathbb{Z}} U_{(n)}$ is a \mathbb{Z} -graded space such that $\deg u(n) = m - n - 1$ for $u \in U_{(m)}$, $n \in \mathbb{Z}$.

We shall conventionally call L a *local vertex Lie algebra over the base space U* .

In the following example, we shall see that affine Lie algebras and the Virasoro algebra are local vertex Lie algebras.

EXAMPLE 3.2. (1) Let $Witt = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L(n)$ be the Witt Lie algebra or the centerless Virasoro algebra, where

$$(3.2) \quad [L(m), L(n)] = (m - n)L(m + n) \quad \text{for } m, n \in \mathbb{Z}.$$

Then $Witt$ is a local vertex Lie algebra over the base space $U = \mathbb{C}\omega$ where $\rho(\omega \otimes t^m) = \omega(m + 1) = L(m)$ for $m \in \mathbb{Z}$, the domain of d is 0 and

$$(3.3) \quad [\omega(x), \omega(y)] = \omega^{(1)}(y)\Delta(x, y) - 2\omega(y)\Delta^{(1)}(x, y).$$

(2) The Virasoro algebra $Vir = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L(n) \oplus \mathbb{C}c$ is a local vertex Lie algebra over the base space $U = \mathbb{C}\omega \oplus \mathbb{C}c$, where

$$\rho(\omega \otimes t^m) = \omega(m + 1) = L(m), \quad \rho(c \otimes t^m) = c(m) = \delta_{m, -1}c$$

for $m \in \mathbb{Z}$, the domain of d is $\mathbb{C}c$ with $d = 0$ and

$$(3.4) \quad [\omega(x), \omega(y)] = \omega^{(1)}(y)\Delta(x, y) - 2\omega(y)\Delta^{(1)}(x, y) - \frac{1}{12}c(y)\Delta^{(3)}(x, y),$$

$$(3.5) \quad [\omega(x), c(y)] = 0.$$

(3) Let \mathfrak{g} be a Lie algebra and let $L(\mathfrak{g})$ be the loop (Lie) algebra

$$(3.6) \quad L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}],$$

where

$$(3.7) \quad [a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} \quad \text{for } a, b \in \mathfrak{g}, m, n \in \mathbb{Z}.$$

The loop algebra $L(\mathfrak{g})$ is a local vertex Lie algebra over the base space \mathfrak{g} with ρ being the identity map and the domain of d being zero, where

$$(3.8) \quad [a(x), b(y)] = [a, b](y)\Delta(x, y) \quad \text{for } a, b \in \mathfrak{g}.$$

(4) If \mathfrak{g} is a Lie algebra with a symmetric nondegenerate invariant bilinear form $(\cdot | \cdot)$, then the corresponding affine Lie algebra $\hat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}c$ is a local vertex Lie algebra over the base space $U = \mathfrak{g} \oplus \mathbb{C}c$ where

$$\rho(a \otimes t^m) = a \otimes t^m, \quad \rho(c \otimes t^m) = \delta_{m, -1}c \quad \text{for } a \in \mathfrak{g}, m \in \mathbb{Z},$$

the domain of d is $\mathbb{C}c$ with $d = 0$, and

$$(3.9) \quad [a(x), b(y)] = [a, b](y)\Delta(x, y) - (a | b)c(y)\Delta^{(1)}(x, y),$$

$$(3.10) \quad [a(x), c(y)] = 0 \quad \text{for } a, b \in \mathfrak{g}.$$

It is not hard to see that all the local vertex Lie algebras discussed here are graded.

REMARK 3.3. (1) Let (L, U, d, ρ) be a local vertex Lie algebra. Then $L \simeq L(U)/\ker \rho$. Clearly, $u \otimes t^{-1} \notin \text{im } d = \ker \rho$ for any $0 \neq u \in U$. Thus $u(-1) \neq 0$ and $u(x) \neq 0$. Then the linear map d is uniquely determined by the condition $\frac{d}{dx}a(x) = (da)(x)$. Set $U^0 = \ker d$. Then $U^0 \otimes \frac{d}{dt}\mathbb{C}[t, t^{-1}] \subset \text{im } d = \ker \rho$.

(2) The operator d implies the following: In the definition of local vertex Lie algebra, if $f_i(u, v), df_i(u, v), \dots, d^{k_i}f_i(u, v)$ are defined then $f_i(u, v)^{(k_i)}(y)$ can be replaced by $(d^{k_i}f_i(u, v))(y)$. In particular, if d is defined on whole U , we can choose $f_i(u, v)$ so that $k_i = 0$ for all i .

The following are consequences of the definitions.

PROPOSITION 3.4. (1) Let (L, U, d, ρ) be a local vertex Lie algebra with $\ker d = U^0$. Then

$$(3.11) \quad u(n) = 0 \quad \text{for } u \in U^0, \quad n \neq -1,$$

and $u(-1)$ for $u \in U^0$ form a central subalgebra of L .

(2) The components of $u(m)$ and $v(n)$ of $u(x_1)$ and $v(x_2)$ have bracket

$$(3.12) \quad [u(m), v(n)] = \sum_{i=0}^r \binom{m}{l_i} \binom{m+n-l_i}{k_i} (-1)^{l_i+k_i} l_i! k_i! f_i(u, v) (m+n-l_i-k_i).$$

(3) Set $L^- = \rho(U \otimes t^{-1}\mathbb{C}[t^{-1}])$, $L^+ = \rho(U \otimes \mathbb{C}[t])$ and $L^0 = \rho(U)$ (a subspace of L^+). Then L^\pm and L^0 are Lie subalgebras of L and $L = L^+ \oplus L^-$ is a polar decomposition. Furthermore, $L^0 = U/(U^0 + \text{im } d)$ and ρ is a linear isomorphism from $(U^0)' \otimes t^{-1} \oplus U' \otimes \mathbb{C}[t, t^{-1}]$ onto L , where U' is a subspace of U such that $U = (U^0 + \text{im } d) \oplus U'$ and $(U^0)'$ is a subspace of U^0 such that $(U^0)' \cap \text{im } d = 0$. If $\ker d \cap \text{im } d = 0$, then we may take $(U^0)' = U^0$.

(4) If $L = \bigoplus_{n \in \mathbb{Z}} L_{(n)}$ is a graded local vertex Lie algebra, we have a triangular decomposition: $L = L_+ \oplus L_0 \oplus L_-$, where $L_\pm = \bigoplus_{n=1}^\infty L_{(\pm n)}$ and $L_0 = L_{(0)}$.

PROOF. Since $\ker \rho = \text{im } \hat{d}$, for $u \in U$, $u = u \otimes t^0 \in \ker \rho$ if and only if

$$(3.13) \quad u \otimes t^0 = dv \otimes t^n + nv \otimes t^{n-1}$$

for some $v \in U$, $n \in \mathbb{Z}$. Clearly, (3.13) is equivalent to that either $u \otimes t^0 = dv \otimes t^n$, $n = 0$, or $dv = 0$ and $u \otimes t^0 = nv \otimes t^{n-1}$. Then $u = u \otimes t^0 \in \ker \rho$ if and only if $u \in U^0 + \text{im } d$. Thus $L^0 = U/(U^0 + \text{im } d)$. From Remark 3.3 (1), we have

$$(3.14) \quad (du)(m) = -mu(m-1) \quad \text{for } u \in U, \quad m \in \mathbb{Z},$$

assuming that u is in the domain of d . Then, for $u \in U^0$, $u(m) = 0$ for $m \neq -1$, i.e., $u(x) = u(-1) \in L[x, x^{-1}]$. It follows from Remark 2.7 that $u(-1)$ commutes with $v(x)$ for any $v \in U$. This proves (1). (2) is immediate from (3.1) by considering the coefficients of $x^{-m-1}y^{-n-1}$. The first part of (3) follows from (2) by noticing that $\binom{m}{l_i} \binom{m+n-l_i}{k_i} = 0$ if $m+n-l_i-k_i < 0$ and $m, n \geq 0$. From (3.14), we have

$$\rho(U^0 \otimes t^{-1} + U' \otimes \mathbb{C}[t, t^{-1}]) = L.$$

From a simple fact in linear algebra, it suffices to prove

$$\ker \rho \cap (U' \otimes \mathbb{C}[t, t^{-1}]) = 0,$$

that is,

$$\text{im } \hat{d} \cap (U' \otimes \mathbb{C}[t, t^{-1}]) = 0.$$

Otherwise, there exist $0 \neq u^i \in U'$, $m_1 > \cdots > m_r$ such that

$$u^1 \otimes t^{m_1} + \cdots + u^r \otimes t^{m_r} \in \text{im } \hat{d}.$$

Then

$$(3.15) \quad u^1 \otimes t^{m_1} + \cdots + u^r \otimes t^{m_r} = \sum_{j=1}^s ((dv^j \otimes t^{n_j} + n_j v^j \otimes t^{n_j-1}),$$

where $v^j \in U$, $n_1 > \cdots > n_s$ and $dv^1 \otimes t^{n_1} + n_1 v^1 \otimes t^{n_1-1} \neq 0$. Compare the highest powers of t on both sides of (3.15). If $dv^1 \neq 0$, we have

$$u^1 \otimes t^{m_1} = dv^1 \otimes t^{n_1},$$

that is, $u^1 = dv^1 \in \text{im } d$. This is a contradiction because $0 \neq u^1 \in U' \cap \text{im } d = 0$. Assume $dv^1 = 0$. Then

$$u^1 \otimes t^{m_1} = n_1 v^1 \otimes t^{n_1-1}$$

if $n_2 \neq m_1 - 1$ and

$$u^1 \otimes t^{m_1} = n_1 v^1 \otimes t^{n_1-1} + dv^2 \otimes t^{n_2}$$

if $n_2 = m_1 - 1$. In both cases, we have $u^1 \in U^0 + \text{im } d$, noting that $v^1 \in \ker d = U^0$. It is also a contradiction.

If $U^0 \cap \text{im } d = \ker d \cap \text{im } d = 0$, using a similar argument we can prove that ρ is a linear isomorphism from $(U^0 \otimes t^{-1}) \oplus (U' \otimes \mathbb{C}[t, t^{-1}])$ onto L .

(4) is obvious. \square

Now we turn our attention to a special class of local vertex Lie algebras - local vertex Poisson algebras. First we recall the definition of Poisson algebras, which is needed in the definition of vertex Poisson algebras.

DEFINITION 3.5. A *Poisson algebra* is a commutative associative algebra A equipped with a Lie algebra structure on A satisfying the following Leibniz rule:

$$(3.16) \quad [a, bc] = [a, b]c + [a, c]b \quad \text{for } a, b, c \in A.$$

A *Poisson ideal* for a Poisson algebra A is an ideal for both the associative algebra A and the Lie algebra $(A, [\cdot, \cdot])$.

Formula (3.16) is equivalent to that for $a \in V$, the operator $[a, \cdot]$ is a derivation of the associative algebra A .

Here is an example of Poisson algebras constructed from a Lie algebra.

EXAMPLE 3.6. Let \mathfrak{g} be a Lie algebra. Then $S(\mathfrak{g})$, the symmetric algebra over \mathfrak{g} , could be considered as the algebra of polynomial functions on \mathfrak{g}^* , where \mathfrak{g}^* is considered as a manifold. It is well-known (cf. [W]) that $S(\mathfrak{g})$ has a Poisson algebra structure. More specifically, let u^i ($i \in I$) be a basis of \mathfrak{g} . Then we may identify $S(\mathfrak{g})$ with $\mathbb{C}[u^i \mid i \in I]$. For $f, g \in \mathbb{C}[u^i \mid i \in I]$, we define

$$(3.17) \quad \{f, g\} = \sum_{i, j \in I} \frac{\partial f}{\partial u^i} \frac{\partial g}{\partial u^j} [u^i, u^j].$$

Then $S(\mathfrak{g})$ becomes a Poisson algebra. In general, for a manifold M , a Poisson algebra structure on M (which means a Poisson algebra structure on $F(M)$, the space of smooth functions on M) amounts to a symplectic structure on M .

Let (L, A, d, ρ) be a local vertex Lie algebra over a commutative associative algebra A . Let $A(x) = \{a(x) \mid a \in A\}$. Then $A \rightarrow A(x)$ via $a \mapsto a(x)$ is a linear isomorphism by Remark 3.3. So $A(x)$ is a commutative associative algebra under product

$$a(x) \circ b(x) = ab(x)$$

for $a, b \in A$. Set

$$(3.18) \quad R(x, y) = \sum_{k \in \mathbb{N}} A(x) \Delta^{(k)}(x, y) \subset A[[x, y, x^{-1}, y^{-1}]].$$

Since A has a unit 1 by assumption, for $k \in \mathbb{N}$ we may identify $\Delta^{(k)}(x, y)$ with $1 \cdot \Delta^{(k)}(x, y)$, an element of $R(x, y)$. Define an action of $A(x)$ on $R(x, y)$ by

$$(3.19) \quad a(x) \circ \sum_{k \in \mathbb{N}} b_k(x) \Delta^{(k)}(x, y) = \sum_{k \in \mathbb{N}} (ab_k)(x) \Delta^{(k)}(x, y).$$

In view of Lemma 2.2, the action is well defined and $R(x, y)$ is a free $A(x)$ -module with a basis $\{\Delta^{(k)}(x, y) \mid k \in \mathbb{N}\}$.

DEFINITION 3.7. 1) A local vertex Lie algebra (L, A, d, ρ) is called a *local vertex Poisson algebra* if A is a unital commutative associative algebra such that for $a, b \in A$,

$$(3.20) \quad \{a(x), b(y)\} = \sum_i f_i(a, b)(y) \Delta^{(i)}(x, y),$$

where $f_i(a, b) \in A$ and that

$$(3.21) \quad \{a(x), bc(y)\} = \{a(x), b(y)\} \circ c(y) + \{a(x), c(y)\} \circ b(y)$$

for $a, b, c \in A$.

(2) A local vertex Poisson algebra (L, A, D, ρ) is called a *local vertex Poisson differential algebra* if D is a derivation of the associative algebra A .

REMARK 3.8. $\{, \}$ in the definition of local vertex Poisson algebra is called a vertex Poisson bracket on A . In the following when we talk about a vertex Poisson bracket on A we always mean a vertex Poisson bracket on A associated to a local vertex Poisson algebra over A .

Here are some examples of local vertex Poisson algebras.

EXAMPLE 3.9. Let A be a Poisson algebra. Set $L(A) = A \otimes \mathbb{C}[t, t^{-1}]$. Endow $L(A)$ with the loop Lie algebra structure of the Lie algebra $(A, \{, \cdot\})$. From Example 3.2 (3), $(L(A), A, d, \rho)$ is a local vertex Lie algebra with the domain of d being 0, ρ being the identity map and

$$\{a(x), b(y)\} = \{a, b\}(y) \Delta(x, y) \quad \text{for } a, b \in A.$$

Let $a, b, c \in A$. We have

$$\begin{aligned} & \{a(x), (bc)(y)\} = \{a, bc\}(y) \Delta(x, y) \\ &= (b\{a, c\})(y) \Delta(x, y) + (\{a, b\}c)(y) \Delta(x, y) \\ &= b(y) \circ \{a, c\}(y) \Delta(x, y) + (\{a, b\}(y) \Delta(x, y)) \circ c(y) \\ (3.22) \quad &= b(y) \circ \{a(x), c(y)\} + \{a(x), b(y)\} \circ c(y). \end{aligned}$$

Thus, $(L(A), A, d, \rho)$ is a local vertex Poisson Lie algebra.

EXAMPLE 3.10. Let \mathfrak{g} be a Lie algebra and let $S(\mathfrak{g})$ be the symmetric algebra on \mathfrak{g} . Let u^i ($i \in I$) be a basis of \mathfrak{g} . Then $S(\mathfrak{g})$ can be identified as the polynomial algebra $\mathbb{C}[u^i \mid i \in I]$ (see Example 3.6). By Examples 3.6, 3.2 (3) and 3.9, $L(S(\mathfrak{g}))$ is a local vertex Poisson algebra over $S(\mathfrak{g})$. In particular, for $f, g \in \mathbb{C}[u^i \mid i \in I]$ we have

$$(3.23) \{f(x), g(y)\} = \{f, g\}(y)y^{-1}\delta\left(\frac{x}{y}\right) = \left(\sum_{i,j \in I} \frac{\partial f}{\partial u^i} \frac{\partial g}{\partial u^j} [u^i, u^j]\right)(y)y^{-1}\delta\left(\frac{x}{y}\right).$$

This vertex Poisson bracket on $S(\mathfrak{g})$ is usually called the *ultra-Poisson* bracket.

PROPOSITION 3.11. *Let (L, A, D, ρ) be a local vertex Poisson differential algebra over A . Then for any $a, b \in A$, $i \in \mathbb{N}$, there exists a unique $a_i b \in A$ such that*

$$(3.24) \quad [a(x), b(y)] = \sum_{i=0}^{\infty} \frac{1}{i!} (a_i b)(y) \Delta^{(i)}(x, y)$$

or equivalently

$$(3.25) \quad [a(m), b(n)] = \sum_{i=0}^{\infty} \binom{m}{i} (a_i b)(m+n-i) \quad \text{for } m, n \in \mathbb{Z}.$$

Moreover $L^0 = A/DA$ is a subalgebra of L with $[a + DA, b + DA] = a_0 b + DA$ for $a, b \in A$.

PROOF. Note that $(Da)(x) = \frac{d}{dx}a(x)$ for any $a \in A$. Then in (3.1) we can replace $f_i(u, v)^{(k_i)}(y)$ by $D^{k_i}f_i(u, v)$. For any $a, b \in A$, $i \in \mathbb{N}$, by Lemma 2.2 we may define $a_i b \in A$ by requiring

$$(3.26) \quad [a(x), b(y)] = \sum_{i=0}^{\infty} \frac{1}{i!} (a_i b)(y) \Delta^{(i)}(x, y).$$

Recall that $a(0) = \rho(a)$ and $L^0 = \rho(A)$ which is isomorphic to A/DA linearly. By (3.25), $[a(0), b(0)] = (a_0 b)(0)$ for $a, b \in A$ or equivalently $[a + DA, b + DA] = a_0 b + DA$. \square

Commutation relations (3.24) and (3.25) are exactly the commutator relations for vertex operators in the theory of vertex operator algebras (see [B], [FLM]).

Next we discuss the relation between local vertex Poisson differential algebras and Poisson algebras. In the definition of local vertex Poisson differential algebra A is only assumed to be a commutative associative algebra. From Proposition 3.11, A/DA has a Lie algebra structure already. If DA is also an ideal of A then A/DA will be a Poisson algebra. But this is not true. It turns out that one has to modulo $A(DA)$ to get a Poisson algebra.

PROPOSITION 3.12. *Let (L, A, D, ρ) be a vertex Poisson differential algebra. Then the quotient space $A/(DA)A$ is a Poisson algebra with the original associative multiplication and the following Lie bracket:*

$$(3.27) \quad [a + (DA)A, b + (DA)A] = a_0 b + (DA)A \quad \text{for } a, b \in A.$$

PROOF. It is clear that $(DA)A$ is an two-sided ideal of A , so that $A/(DA)A$ is a commutative associative algebra. By Proposition 3.11, $A/D(A)$ is a Lie algebra with the Lie bracket: $[a + D(A), b + D(A)] = a_0b + D(A)$ for $a, b \in A$. Taking $\text{Res}_x \text{Res}_y$ of (3.21) then gives

$$(3.28) \quad [ab, c] = a[b, c] + b[a, c] \quad \text{for } a, b, c \in A.$$

Then $(DA)A/DA$ is a Lie ideal of A/DA so that $A/(DA)A$ is a Lie algebra with the Lie bracket, and the Leibniz rule is clear. The proof is complete. \square

Let A be a polynomial algebra in variables $u_i^{(j)}$ for $i \in I$, $j \in \mathbb{N}$, where I is an index set. Then A has a derivation uniquely determined by $\partial u_i^{(j)} = u_i^{(j+1)}$. It is clear that any vertex Poisson bracket on A is uniquely determined by $\{u_i(x), u_j(y)\}$ for $i, j \in I$.

DEFINITION 3.13. A *differential-geometric* Poisson bracket on A (see [DB]) is a vertex Poisson bracket $\{, \}$ on A such that

$$(3.29) \quad \{u_i(x), u_j(y)\} = \sum_{l=0}^{n(i,j)} \psi_{ij}^l(u)(x) \Delta^{(l)}(x, y)$$

for any $i, j \in I$, $n(i, j) \in \mathbb{N}$, where $\psi_{ij}^l(u) \in A$. A Poisson bracket on A is said to be *hydrodynamic type* if it has the form

$$(3.30) \quad \{u^i(x), u^j(y)\} = g^{ij}(u(x)) \Delta^{(1)}(x, y) + \sum_{k \in I} b_k^{ij}(u(x)) u_k^{(1)} \Delta(x, y).$$

REMARK 3.14. In Definition 3.13, if each $\psi_{ij}^k(u)$ is a linear function in $u_j^{(l)}$ for $j \in I$, $l \in \mathbb{N}$, then one obtains a (smaller) local vertex Lie algebra over the space U with a basis $\{u_i \mid i \in I\}$.

The following interesting theorem was due to [DN]:

THEOREM 3.15. (1) Under local changes of the fields $u = u(w)$ the coefficient g^{ij} in the bracket (3.30) transforms like a bilinear form (a tensor with upper indices); if $\det g^{ij} \neq 0$, then the expression $b_k^{ij} = g^{is} \Gamma_{sk}^j$ transforms in such a way that the Γ_{sk}^j are the Christoffel symbols of a differential-geometric connection.

(2) In order that the bracket (3.30) be skew-symmetric it is necessary and sufficient that the tensor $g^{ij}(u)$ be symmetric (i.e., that it defines a pseudo-Riemannian metric if $\det g^{ij} \neq 0$) and the connection Γ_{sk}^j be consistent with the metric, $g_k^{ij} = \Delta_k g^{ij} = 0$.

(3) In order that the bracket (3.30) satisfy the Jacobi identity it is necessary and sufficient that the connection Γ_{sk}^j has no torsion and the curvature tensor vanish. In this case the connection is defined by the metric $g^{ij}(u)$ which can be reduced to constant form.

We discuss more examples:

EXAMPLE 3.16. Suppose that $\{u_i(x), u_j(y)\} = d_{ij} \Delta^{(1)}(x, y)$ for $i, j \in I$, where $d_{ij} \in \mathbb{C}$. Set $H = \oplus_{i \in I} \mathbb{C} u_i$. Then define $(u_i | u_j) = d_{ij}$ for $i, j \in I$. Set $\hat{H} = H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c$. Then a vertex Poisson bracket of this type determines a Lie algebra structure on \hat{H} with $[u_i(m), u_j(n)] = d_{ij} m \delta_{m+n, 0} c$ and $[\hat{H}, c] = 0$. That is,

a vertex Poisson bracket of this type determines a Heisenberg Lie algebra structure on \hat{H} .

More generally, suppose

$$(3.31) \quad \{u_i(x), u_j(y)\} = d_{ij}\Delta^{(1)}(x, y) + \sum_{k \in I} c_{ij}^k u_k(x)\Delta(x, y)$$

for $i, j \in I$, where d_{ij} and c_{ij}^k are constants. Then a vertex Poisson algebra of this type gives rise to a Lie algebra structure on the subspace \mathfrak{g} with a basis $\{u_1, u_2, \dots\}$ where

$$[u_i, u_j] = \sum_{k \in I} c_{ij}^k u_k \quad \text{for } i, j \in I$$

and an affine Lie algebra structure on $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$ with the standard bracket formula.

On the other hand, let \mathfrak{g} be a Lie algebra with a nondegenerate symmetric invariant bilinear form (\cdot, \cdot) . Let $\{u_i \mid i \in I\}$ be a basis of \mathfrak{g} such that $(u_i, u_j) = d_{ij}$ and

$$[u_i, u_j] = \sum_k c_{ij}^k u_k.$$

Then the corresponding affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}$ has bracket

$$\{u_i(x), u_j(y)\} = d_{ij}\Delta^{(1)}(x, y) + \sum_{k \in I} c_{ij}^k u_k(x)\Delta(x, y)$$

where $u_i(x) = \sum_{n \in \mathbb{Z}} (u_i \otimes t^n) x^{-n-1}$. Note also that $u(n) = u \otimes t^n$ in our notation.

EXAMPLE 3.17. Consider a vertex Poisson Lie algebra structure on A such that

$$(3.32) \quad \{u_i(x), u_j(y)\} = \sum_{k \in I} d_{ij}^k u_k^{(1)}(y)\Delta(x, y) + \sum_{k \in I} c_{ij}^k u_k(y)\Delta^{(1)}(x, y).$$

If we define an algebra B with a basis $\{a_i \mid i \in I\}$ such that $a_i a_j = \sum_{k \in I} c_{ij}^k a_k$, then B is commutative and associative [Za]. In this case, we have a local vertex Lie algebra over B where

$$(3.33) \quad [a(m), b(n)] = \frac{1}{2}(m-n)(ab)(m+n-1) \quad \text{for } a, b \in B, m, n \in \mathbb{Z}.$$

In [BN], essentially the same result was obtained by considering

$$(3.34) \quad \{u^i(x), u^j(y)\} = \sum_{k \in I} d_{ij}^k u_k^{(1)}(x)\Delta(x, y) + \sum_{k \in I} c_{ij}^k u_k(x)\Delta^{(1)}(x, y),$$

where $d_{ij}^k, c_{ij}^k \in \mathbb{C}$ for $i, j, k \in I$. Notice that using (2.11) we may exchange (3.34) with (3.32). The same result was also obtained in [GD].

One can also consider a vertex Lie algebra over $B \oplus \mathbb{C}$ with

$$(3.35) \quad \{u^i(x), u^j(y)\} = \sum_{k \in I} d_{ij}^k u_k^{(1)}(y)\Delta(x, y) + \sum_{k \in I} c_{ij}^k u_k(y)\Delta^{(1)}(x, y) + f_{ij}\Delta^{(3)}(x, y),$$

where $d_{ij}^k, c_{ij}^k, f_{ij} \in \mathbb{C}$ for $i, j, k \in I$. Let B be defined as before and define a bilinear form $(\cdot | \cdot)$ on B such that $(u_i | u_j) = f_{ij}$ for $i, j \in I$. Then

$$(3.36) \quad [a(m), b(n)] = \frac{1}{2}(m-n)(ab)(m+n-1) + \frac{1}{6}(a|b)(m^3 - m)\delta_{m, -n}c$$

for $a, b \in B$, $m, n \in \mathbb{Z}$. It was proved in [La] that (3.36) defines a Lie algebra structure if and only if B is commutative and associative, and $(\cdot|\cdot)$ is a symmetric associative form on B . (Lam obtained this result from a different point of view by considering the weight 2 subspace of certain vertex operator algebras.)

EXAMPLE 3.18. Consider a vertex Poisson algebra structure on A such that

$$(3.37) \quad \{u_i(x), u_j(y)\} = g^{ij}(x)\Delta^{(2)}(x, y) + \sum_{k \in I} b_k^{ij}(x)u_k^{(1)}(x)\Delta^{(1)}(x, y).$$

Then it was proved in [Po] that the coefficients g^{ij} and b_k^{ij} must satisfy the following relations:

$$(3.38) \quad g^{ij} = -g^{ji}, \quad \frac{\partial g^{ij}}{\partial u_k} = b_k^{ij}, \quad \sum_{l \in I} b_l^{ij} g^{lk} = \sum_{l \in I} b_l^{jk} g^{li};$$

$$(3.39) \quad \sum_{l \in I} \frac{\partial (b_l^{ij} g^{lk})}{\partial u_m} = \sum_{l \in I} \left(b_l^{ij} b_m^{lk} + b_l^{jk} b_m^{li} + b_l^{ki} b_m^{lj} \right).$$

Furthermore, suppose that the coefficient b_k^{ij} does not depend on u_i , then $g^{ij} = \sum_{k \in I} b_k^{ij} u_k + g_0^{ij}$, where b_k^{ij} are the structural constants of an associative anticommutative algebra B that satisfy the condition

$$(3.40) \quad \sum_{l \in I} b_l^{ij} g_0^{lk} = \sum_{l \in I} b_l^{jk} g_0^{li}.$$

Define a bilinear form on B such that $(u_i, u_j) = g_0^{ij}$. Then we have

$$(3.41) \quad (u_i u_j, u_k) = \sum_{l \in I} b_l^{ij} g_0^{lk} = \sum_{l \in I} b_l^{jk} g_0^{li} = (u_j u_k, u_i).$$

Thus $(ab, c) = (bc, a)$ for $a, b, c \in B$. Using (2.11) we have

$$(3.42) \quad \begin{aligned} \{u_i(x), u_j(y)\} &= \sum_{k \in I} \left(-b_{ij}^k u_k^{(1)}(y) \Delta^{(1)}(x, y) + (b_{ij}^k u_k(y) + g_0^{ij}) \Delta^{(2)}(x, y) \right) \\ &= -(u_i u_j)^{(1)}(y) \Delta^{(1)}(x, y) + ((u_i u_j)(y) + (u_i | u_j)) \Delta^{(2)}(x, y). \end{aligned}$$

Thus

$$(3.43) \quad \begin{aligned} &[a(m), b(n)] \\ &= -m(m+n-1)(ab)(m+n-2) \\ &\quad + m(m-1)((ab)(m+n-2) + (a|b)\delta_{m+n,1}) \\ &= -mn(ab)(m+n-2) + m(m-1)\delta_{m+n,1}(a|b) \end{aligned}$$

for $a, b \in B$, $m, n \in \mathbb{Z}$ and $[\hat{B}, \mathbb{C}] = 0$. One can easily show that \hat{B} is a Lie algebra if and only if $B^3 = 0$.

4. Constructing vertex algebras from a local vertex Lie algebra

The main goal of this section is to give a construction of vertex algebras from a local vertex Lie algebra. Roughly speaking, we shall construct quantum objects, vertex algebras, from a classical object, a local vertex Lie algebra. Results similar to some of ours have been also obtained in [K2], [MP], [Pr], [R] and [X].

First, let us review the definitions of vertex (operator) algebra and module from [B] and [FLM] (cf. [Li1]).

DEFINITION 4.1. A vertex algebra is a triple $(V, Y, \mathbf{1})$ consisting of a vector space V , a vector $\mathbf{1} \in V$ and linear map Y from V to $(\text{End } V)[[x, x^{-1}]]$ satisfying the following axioms:

- (1) $Y(u, x)v \in V((x))$ for $u, v \in V$,
- (2) $Y(\mathbf{1}, x) = 1$,
- (3) $Y(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{x \rightarrow 0} Y(u, x)\mathbf{1} = u$ for $u \in V$,
- (4) The Jacobi identity holds for $u, v \in V$:

$$(4.1) \quad \begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2). \end{aligned}$$

We shall also use V for the vertex algebra $(V, Y, \mathbf{1})$.

DEFINITION 4.2. Let V be a vertex algebra. A V -module is a vector space W equipped with a linear map Y_W from V to $(\text{End } W)[[x, x^{-1}]]$ satisfying the following axioms:

- (1) $Y_W(u, x)w \in W((x))$ for $u \in V, w \in W$,
- (2) $Y_W(\mathbf{1}, x) = 1$,
- (3) The Jacobi identity holds for $u, v \in V$:

$$(4.2) \quad \begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2) Y_W(u, x_1) \\ &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2). \end{aligned}$$

The notions of submodule and homomorphism can be defined accordingly.

We shall need the following result from [Li1].

PROPOSITION 4.3. *Let V be a vertex algebra, W a V -module and u a vector in W such that $a_n u = 0$ for any $a \in V, n \in \mathbb{N}$. Then the linear map ψ from V to W defined by $\psi(a) = a_{-1}u$ for $a \in V$ is a V -homomorphism.*

A V -module W is said to be *faithful* if the vertex operator map Y_W is injective. The following result was proved in [Li2].

PROPOSITION 4.4. *Let V be a vertex algebra, W a faithful V -module. Let $a, b, c^{(0)}, \dots, c^{(k)} \in V$. Then*

$$[Y(a, x), Y(b, y)] = \sum_{j=0}^k \frac{(-1)^j}{j!} Y(c^{(j)}, y) \left(\frac{\partial}{\partial x} \right)^j y^{-1} \delta \left(\frac{x}{y} \right)$$

if and only if

$$[Y_W(a, x), Y_W(b, y)] = \sum_{j=0}^k \frac{(-1)^j}{j!} Y_W(c^{(j)}, y) \left(\frac{\partial}{\partial x} \right)^j y^{-1} \delta \left(\frac{x}{y} \right).$$

In [Li2], a general machinery has been built to produce vertex algebras by using the notion of so-called local system of vertex operators. Next we shall recall some results about local systems.

Let M be a vector space. A *vertex operator* on M is a formal series $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (\text{End } M)[[z, z^{-1}]]$ such that for any $u \in M, a_n u = 0$ for sufficiently large n . All vertex operators on M form a vector space (over \mathbb{C}), denoted

by $VO(M)$. On $VO(M)$, we have a linear endomorphism $D = \frac{d}{dx}$, the formal differentiation.

Two vertex operators $a(z)$ and $b(z)$ on M are said to be *mutually local* if there is a non-negative integer k such that

$$(4.3) \quad (z_1 - z_2)^k a(z_1)b(z_2) = (z_1 - z_2)^k b(z_2)a(z_1).$$

A space S of vertex operators is said to be *local* if any two vertex operators of S are mutually local, and a maximal local space of vertex operators is called a *local system*.

Let V be a local system on M . Then V is closed under the formal differentiation $D = \frac{d}{dx}$. For $a(x), b(x) \in VO(M)$, we define

$$(4.4) \quad \begin{aligned} & Y(a(x), z)b(x) \\ &= \text{Res}_{z_1} \left(z^{-1} \delta \left(\frac{z_1 - x}{z} \right) a(z_1)b(x) - z^{-1} \delta \left(\frac{x - z_1}{-z} \right) b(x)a(z_1) \right). \end{aligned}$$

Denote by $I(x)$ the identity endomorphism of M .

THEOREM 4.5. *Let M be a vector space and V a local system on M . Then $(V, Y, I(x))$ is a vertex algebra with M as a natural module such that $Y_M(a(x), z) = a(z)$ for $a(x) \in V$.*

Let A be any local space of vertex operators on M . Then there exists a local system V that contains A . Let $\langle A \rangle$ be the vertex subalgebra of V generated by A . Since the vertex operator “product” (4.4) does not depend on the choice of local system V , $\langle A \rangle$ is canonical. Then we have:

COROLLARY 4.6. *Let M be a vector space and A any local space of vertex operators on M . Then A generates a canonical vertex algebra $\langle A \rangle$ with M as a natural module such that $Y_M(a(x), z) = a(z)$ for $a(x) \in A$.*

Next we should construct vertex algebras from any local vertex Lie algebra L .

PROPOSITION 4.7. *Let L be a local vertex Lie algebra over U , let M be a restricted L -module and let V be any vertex algebra of vertex operators on M containing all $u(x)$ for $u \in U$. Then V is an L -module with $u(n)$ acting on V by $u(x)_n$ for $u \in U$, $n \in \mathbb{Z}$.*

PROOF. For $u, v \in U$, suppose

$$[u(x), v(y)] = \sum_{j=0}^n \Delta^{(j)}(x, y) w_j(y),$$

where $w_j(y)$ lies in the space spanned by $w^{(i)}(y)$ for $w \in U$ and $i \geq 0$. Then V is an L -module if and only if

$$(4.5) \quad [Y(u(x), z_1), Y(v(x), z_2)] = \sum_{j=0}^n \Delta^{(j)}(z_1, z_2) Y(w_j(x), z_2).$$

Since M is an L -module, we have

$$[\bar{u}(x), \bar{v}(y)] = \sum_{j=0}^n \Delta^{(j)}(x, y) \bar{w}_j(y),$$

where $\bar{u}(x) = \sum_{m \in \mathbb{Z}} \bar{u}(n)$ and $\bar{u}(n)$ is an element of $\text{End} M$ representing $u(n)$. Since M is a faithful V -module, (4.5) follows from Proposition 4.4. This concludes the proof. \square

Let L be a local vertex Lie algebra over U . Recall from Proposition 3.4 (3) that L has a polar decomposition $L = L^+ \oplus L^-$ where L^\pm are Lie subalgebras of L . Consider the induced L -module $V(L) = U(L) \otimes_{U(L^+)} \mathbb{C}$ where \mathbb{C} is the one-dimensional trivial L^+ -module. Set $\mathbf{1} = 1 \otimes 1$. Then we have ([Li1], see also [K2], [FKRW], [MP], [Pr], [R], [X]):

THEOREM 4.8. *Let L be any local vertex Lie algebra over U . Then $V(L)$ has a vertex algebra structure with any restricted L -module as a natural module.*

PROOF. Let M be any restricted L -module. Then $W = V(L) + M$ is a restricted L -module. Set $\bar{U} = \{u(x) \mid u \in U\}$. Then \bar{U} is a local subspace of vertex operators on W . Thus \bar{U} generates a canonical vertex algebra V such that W is a natural V -module. In particular, both $V(L)$ and M are V -modules. If we can prove that $V \simeq V(L)$, then we are done. It is clear that V is an L -module generated by L from Id_V by Proposition 4.7. From the axioms of vertex algebra we know that $u(n)\mathbf{1} = u(x)_n \mathbf{1} = 0$ for all $u \in U$ and $n \geq 0$. Thus V is a quotient L -module of $V(L)$.

On the other hand, $u(x)_n \mathbf{1} = u(n)\mathbf{1} = 0$ for all $u \in U$ and $n \geq 0$. It is easy to prove (cf. [GL]) that $a_n \mathbf{1} = 0$ for all $v \in V$ and $n \geq 0$. Then by Proposition 4.3, the linear map ψ from V to $V(L)$ defined by $f(a) = a_{-1}I(x)$ is a V -homomorphism. In particular ψ is an L -homomorphism. Thus ψ is an L -isomorphism. The proof is complete. \square

Let L be a local vertex Lie algebra over U . Recall from Proposition 4.7 that U^0 is the kernel of d and $U^0(-1) = \{u(-1) \mid u \in U^0\}$ is a central subalgebra of L . Let λ be any linear character of $U^0(-1)$. Then we define $V(L, \lambda)$ to be the quotient module of $V(L)$ modulo the relation $c \cdot \mathbf{1} = \lambda(c)$ for $c \in U^0(-1)$. As a simple consequence we have:

COROLLARY 4.9. *Let L be a local vertex Lie algebra over U and λ a linear character of U^0 . Then $V(L, \lambda)$ is a quotient vertex algebra of $V(L)$.*

Suppose that L is a graded local vertex Lie algebra over $U = \bigoplus_{n=0}^{\infty} U(n)$ such that $U(0) = U^0$. The degree of $u \in U(n)$ is defined to be n . Then for any $\lambda \in U(0)^*$, $V(L, \lambda)$ has a natural \mathbb{N} -grading $V(L, \lambda) = \bigoplus_{n \in \mathbb{N}} V(L, \lambda)_{(n)}$ with

$$\deg(u_1(-n_1) \cdots u_k(-n_k)\mathbf{1}) = \deg u_1 + n_1 - 1 + \cdots + \deg u_k + n_k - 1$$

for homogeneous u^i in U of positive degrees and with $V(L, \lambda)_{(0)} = \mathbb{C}\mathbf{1}$. The $V(L, \lambda)$ has a unique maximal graded proper ideal $J(L, \lambda)$ so that $L(L, \lambda) = V(L, \lambda)/J(L, \lambda)$ is a simple graded vertex algebra. If the Virasoro algebra is a Lie subalgebra of L associated with a vector $\omega \in U(2)$ and the \mathbb{N} -grading on $V(L, \lambda)$ is given by the operator $L(0)$, then $L(L, \lambda)$ is a simple vertex operator algebra.

5. Lie algebras and Poisson algebras associated with vertex operator algebras

In this section, we study Poisson algebras and Poisson vertex algebras associated with vertex algebras.

Let $(V, Y, \mathbf{1})$ be a vertex algebra and let D be the endomorphism of V defined by

$$(5.1) \quad D(a) = a_{-2}\mathbf{1} \quad \text{for } a \in V.$$

Set $\hat{V} = V \otimes \mathbb{C}[t, t^{-1}]$ and $\hat{D} = D \otimes 1 + 1 \otimes \frac{d}{dt}$. Then $g(V) = \hat{V}/\hat{D}\hat{V}$ (by definition) is a Lie algebra (cf. [B], [FFR], [Li3]) with bracket

$$(5.2) \quad [a(m), b(n)] = \sum_{i=0}^{\infty} \binom{m}{i} (a_i b)(m+n-i),$$

where $a(m)$ stands for $a \otimes t^m + \hat{D}\hat{V}$.

Notice that in terms of generating functions, (5.2) can be written as

$$(5.3) \quad [a(x), b(y)] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (a_k b)(y) \left(\frac{\partial}{\partial x} \right)^k y^{-1} \delta\left(\frac{x}{y}\right)$$

where $a(x) = \sum_{n \in \mathbb{Z}} a(n)x^{-n-1}$.

We should also mention that any V -module becomes a natural $g(V)$ -module with $a(m)$ acting as a_m for $a \in V$, $m \in \mathbb{Z}$.

REMARK 5.1. Let V be a vertex algebra and let D be the endomorphism of V defined above. Define an onto linear map ρ from \hat{V} to $g(V)$ by sending $u \otimes t^n$ to $u(n)$ for $u \in V$, $n \in \mathbb{Z}$. Then $\ker \rho = \text{im } \hat{D}$. Thus the Lie algebra $g(V)$ is a local vertex Lie algebra over V .

Let V be a vertex algebra. Set

$$(5.4) \quad \hat{V}^- = V \otimes t^{-1}\mathbb{C}[t^{-1}], \quad \hat{V}^+ = V \otimes \mathbb{C}[t], \quad \hat{V}^0 = V \otimes \mathbb{C} (= V).$$

Then \hat{V}^\pm and V are \hat{D} -invariant subspaces. Set $g(V)^\pm = \hat{V}^\pm/\hat{D}\hat{V}^\pm$ and $g(V)^0 = V/DV$. Then it is clear that $g(V)^\pm$ and $g(V)^0$ are subalgebras of $g(V)$ and we have the following polar decomposition

$$(5.5) \quad g(V) = g(V)^+ \oplus g(V)^-.$$

Notice that $\mathbf{1}(n) = 0$ for $n \neq -1$ and that $\mathbf{1}(-1)$ is a central element of $g(V)$. Then for any $g(V)$ -module M and any complex number ℓ , $(\mathbf{1}(-1) - \ell)M$ is a submodule of M .

Let ℓ be any complex number. Let \mathbb{C} be the one-dimensional trivial $(g(V)^+ \oplus g(V)^0)$ -module and consider the induced module

$$(5.6) \quad I(V) = U(g(V)) \otimes_{(g(V)^+ \oplus g(V)^0)} \mathbb{C}.$$

Set

$$(5.7) \quad V^{[\ell]} = I(V)/(\mathbf{1}(-1) - \ell)I(V).$$

Then $V^{[\ell]}$ is a $g(V)$ -module with $\mathbf{1}(-1)$ acting as a scalar ℓ . Let π_ℓ be the linear map from V to $V^{[\ell]}$ defined by $\pi_\ell(a) = a(-1)\mathbf{1}$ for $a \in V$ where $\mathbf{1} = 1 \otimes 1 + (\mathbf{1}(-1) - \ell)I(V)$. (We use the bold faced letter $\mathbf{1}$ for the vacuum vector of V .) Then it follows from the PBW theorem that π_ℓ is injective if $\ell \neq 0$. If $\ell = 1$, it follows from the universal property of $V^{[1]}$ that V is a quotient $g(V)$ -module of $V^{[1]}$.

THEOREM 5.2. *Let V be any vertex algebra and let ℓ be any complex number. Then $V^{[\ell]}$ has a natural vertex algebra structure.*

PROOF. Since $g(V)$ is a local vertex algebra on the base space V , $V(g(V))$ is a vertex algebra. Then it is clear that $V^{[\ell]}$ is a quotient vertex algebra of $V(g(V))$ for any complex number ℓ . \square

Let V be a vertex algebra and let n be a positive integer. Then (see [FHL]) $V^{\otimes n}$ has a natural vertex algebra structure. Then $V^{\otimes n}$ is a natural $g(V)$ -module of level n . Let $V^{(n)}$ be the $g(V)$ -submodule of $V^{\otimes n}$ generated from the vacuum $\mathbf{1}$. It is easy to see that $V^{(n)}$ is a vertex subalgebra of $V^{\otimes n}$. It follows from the universal property of $V^{[n]}$ as a $g(V)$ -module that $V^{(n)}$ is a quotient vertex algebra of $V^{[n]}$.

The following lemma shows that $g(V)^-$ as a vector space is linearly isomorphic to V . (This result was also independently obtained in [Pr].)

LEMMA 5.3. *Let V be any vertex algebra. Then V equipped with the product “ $*$ ” defined by*

$$(5.8) \quad a * b = a_{-1}b \quad \text{for } a, b \in V.$$

is a Lie admissible algebra with identity in the sense that V is a Lie algebra with the Lie bracket

$$(5.9) \quad [a, b] = a * b - b * a = a_{-1}b - b_{-1}a \quad \text{for } a, b \in V.$$

Furthermore, it is isomorphic to $g(V)^-$.

PROOF. Noticing that $(Da)(m) = -ma(m-1)$ for $a \in V$, $m \in \mathbb{Z}$, by using induction on n we obtain $a(-n) = \frac{1}{(n-1)!}(D^{n-1}a)(-1)$ for $a \in V$ and any positive integer n . Because of this relation, $g(V)^-$ is linearly spanned by $a(-1)$ for $a \in V$. Then the linear map f from V to $g(V)$ defined by $f(a) = a(-1)$ for $a \in V$ is onto. Since V is a $g(V)^-$ -module with $a(-n)$ being represented by a_{-n} for $a \in V$, $n = 1, 2, \dots$, all a_{-1} for $a \in V$ form a Lie subalgebra of $gl(V)$. Let π be the representation on V of $g(V)^-$. Then πf is the linear map from V to $\text{End} V$ such that $\pi f(a) = a_{-1}$. Because $a_{-1}\mathbf{1} = a$ for any $a \in V$, πf is injective, so that f is injective. Thus f is a linear isomorphism. For $a, b \in V$, suppose $[a_{-1}, b_{-1}] = c_{-1}$ for some $c \in V$. Then

$$(5.10) \quad c = c_{-1}\mathbf{1} = [a_{-1}, b_{-1}]\mathbf{1} = a_{-1}b_{-1}\mathbf{1} - b_{-1}a_{-1}\mathbf{1} = a_{-1}b - b_{-1}a.$$

Therefore, $[a_{-1}, b_{-1}] = (a_{-1}b - b_{-1}a)_{-1}$. Then the lemma follows immediately. \square

REMARK 5.4. It follows from the proof of Lemma 5.3 that V is a faithful $g(V)^-$ -module.

Suppose that $V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$ is a vertex operator algebra. For $a \in V_{(n)}$, we define $\deg a = \text{wt } a = n$. For $a \in V_{(m)}$, $b \in V_{(n)}$, we have (see [FLM] for example)

$$\text{wt}(a_{-1}b) = \text{wt}(b_{-1}a) = \text{wt } a + \text{wt } b = m + n.$$

Then $V (\simeq g(V)^-)$ becomes a \mathbb{Z} -graded Lie algebra, so that $g(V)^-$ and V are isomorphic as graded vector spaces.

If V is of CFT type in the sense that $V_{(n)} = 0$ for all $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$ (cf. [DLMM]), then it follows from the PBW theorem that $V^{[\ell]}$ is an \mathbb{N} -graded vector space with finite-dimensional homogeneous subspaces.

COROLLARY 5.5. *Let V be a vertex operator algebra of CFT type of rank r and ℓ a complex number. Then $V^{[\ell]}$ is a vertex operator algebra of rank $r\ell$.*

REMARK 5.6. Let G be an automorphism group of a vertex operator algebra V . Then G is also an automorphism group for both the graded nonassociative algebra $(V, *)$ and the graded Lie algebra $(V, [,])$.

LEMMA 5.7. *Let V be a simple vertex operator algebra. Then the center of the Lie algebra or the nonassociative algebra V is $\mathbb{C}\mathbf{1}$.*

PROOF. It was proved in [Li1] that if V is a simple vertex operator algebra, then $\ker L(-1) = \mathbb{C}\mathbf{1}$. Since $(V, *)$ is a graded algebra, the center is also graded. Let $u \in V$ be any homogeneous vector in the center of V . Then $u_{-1}v = v_{-1}u$ for any $v \in V$. By the skew-symmetry we get

$$\begin{aligned} u_{-1}v &= \text{Res}_z z^{-1}Y(u, z)v = \text{Res}_z z^{-1}e^{zL(-1)}Y(v, -z)u \\ (5.11) \quad &= \sum_{i=0}^{\infty} \frac{1}{i!} L(-1)^i (-1)^i v_{i-1}u. \end{aligned}$$

Thus $\sum_{i=1}^{\infty} \frac{1}{i!} (-1)^i L(-1)^i v_{i-1}u = 0$ for any $v \in V$. Since $\ker L(-1) = \mathbb{C}\mathbf{1}$, we obtain

$$\sum_{i=1}^{\infty} \frac{1}{i!} (-1)^i L(-1)^{i-1} v_{i-1}u \in V_{(0)}$$

for any $v \in V$, so that $\text{wt } u + \text{wt } v - 1 = 0$ if $v_j u \neq 0$ for some $j \in \mathbb{N}$. That is, if $\text{wt } u + \text{wt } v \neq 1$, then $v_j u = 0$ for any $j \in \mathbb{N}$. Let k be a positive integer such that $k+1 > -\text{wt } u$, so that $\text{wt } L(-1)^k \omega + \text{wt } u > 1$. Then we have

$$(L(-1)^k \omega)_j u = 0 \quad \text{for any } j \in \mathbb{N}.$$

In particular, $(L(-1)^k \omega)_k u = 0$. Thus $\omega_0 u = L(-1)u = 0$. Then $u \in \mathbb{C}\mathbf{1}$. This concludes the proof. \square

Another subalgebra $g(V)^0$ of $g(V)$ by definition is isomorphic to the quotient space V/DV . If V is a vertex operator algebra, then $g(V)^0$ is a \mathbb{Z} -graded Lie algebra with $\text{dega}(0) = \text{wt } a - 1$ for any homogeneous vector $a \in V$.

REMARK 5.8. For Kac-Moody algebras (see [K1]), one has the standard triangular decomposition where the positive part and the negative part are isomorphic (graded) Lie algebras. If V is a vertex operator algebra, one can get a triangular decomposition $g(V) = g(V)_- \oplus g(V)_0 \oplus g(V)_+$ by defining $\text{dega}(n) = \text{wt } a - n - 1$ for any homogeneous element $a \in V$ and $n \in \mathbb{Z}$. Then one can prove that $g(V)_-$ and $g(V)_+$ are isomorphic Lie algebras by using some of the proof for the contragredient module in [FHL]. However, $g(V)_-$ is not a graded Lie algebra with finite-dimensional homogeneous subspaces. An interesting question is: can we make $g(V)$ a doubly graded Lie algebra with finite-dimensional homogeneous subspaces?

Let V be a vertex algebra as before. Set $P_n(V) = V/C_n(V)$ for $n \geq 2$ where $C_n(V)$ is the subspace of V spanned by $a_{-n}b$ for $a, b \in V$. In the case that V is a vertex operator algebra associated to a highest weight module for the Virasoro algebra or an affine Kac-Moody algebra, $P_n(V)$ has been studied in [FF2] and [FKLMM]. We have (see [Zh]):

PROPOSITION 5.9. *The space $P_2(V)$ is a Poisson algebra with the following associative multiplication \cdot and Lie multiplication $[,]$ defined by*

$$(5.12) \quad (a + C_2(V)) \cdot (b + C_2(V)) = (a_{-1}b + C_2(V)),$$

$$(5.13) \quad [a + C_2(V), b + C_2(V)] = a_0b + C_2(V)$$

for $a, b \in V$.

It follows from Proposition 5.9 that $C_2(V)$ is a two-sided ideal of the Lie admissible algebra $(V, *)$ such that the quotient algebra is commutative and associative.

An obvious question is that to what extent $P_2(V)$ determines V . For example, if V and U are two nonisomorphic vertex algebras, are $P_2(V)$ and $P_2(U)$ nonisomorphic also? Or more loosely, how much information can we get from $P_2(V)$ for the vertex algebra V ? It will be very nice if one can answer these questions.

Next we shall compute $P_2(V(L))$ for a local vertex Lie algebra L . Note that $V(L)$ is spanned by $u^1(-n_1) \cdots u^k(-n_s)\mathbf{1}$ for $s \geq 0$, $u^i \in U$, $n_i > 0$. In view of Remark 3.3 (1), the linear map $u \mapsto u(-1)$ from U to L is injective. Then the linear map $u \mapsto u(-1)\mathbf{1}$ from U to $V(L)$ is injective. Now, we identify U as a subspace of $V(L)$ through the linear map $u \mapsto u(-1)\mathbf{1}$ for $u \in U$.

Let B be the subspace of $V(L)$ linearly spanned by $a(-2-n)V(L)$ for $a \in U$, $n \in \mathbb{N}$. Note that $a(m) = a_m$ for our notations. From the definition of C_2 , we have $B \subset C_2$. In the following we shall show that in fact, $B = C_2$. First, it follows from (3.12) that $b(-1)B \subset B$ for $b \in U$. Suppose $u \in V(L)$ such that $u_{-2-n}V(L) \subset B$ for all $n \geq 0$. Let $a \in U$, $k \in \mathbb{N}$. Then for $n \in \mathbb{N}$ using the Jacobi identity for $V(L)$ we have

$$\begin{aligned}
 & \text{Res}_z z^{-n-2} Y(a(-k-1)u, z) \\
 = & \text{Res}_z z^{-n-2} \text{Res}_{z_1} (z_1 - z)^{-k-1} Y(a, z_1) Y(u, z) \\
 & - \text{Res}_z z^{-n-2} \text{Res}_{z_1} (-z + z_1)^{-k-1} Y(u, z) Y(a, z_1) \\
 = & \text{Res}_z z^{-n-2} \sum_{i=0}^{\infty} \binom{-k-1}{i} ((-z)^i a(-k-1-i) Y(u, z) - (-z)^{-k-1-i} Y(u, z) a(i)) \\
 = & \sum_{i=0}^{\infty} \binom{-k-1}{i} ((-1)^i a(-k-1-i) u_{-n-2+i} - (-1)^{-k-1-i} u_{-n-k-3-i} a(i)) \\
 = & a(-k-1) u_{-n-2} - (-1)^{-k-1} u_{-n-k-3} a(0) \\
 & + \sum_{i \geq 1} \binom{-k-1}{i} ((-1)^i a(-k-1-i) u_{-n-2+i} - (-1)^{-k-1-i} u_{-n-k-3-i} a(i)).
 \end{aligned}$$

Then it follows that $(a(-k-1)u)_{-n-2}V(L) \subset B$. (For $i = 0$, we are using the fact that $b(-1)B \subset B$ and $b(-n-2)V(L) \subset B$ for $b \in U$, $n \in \mathbb{N}$.) Since $V(L)$ is linearly spanned by $u^1(-n_1) \cdots u^s(-n_s)\mathbf{1}$ for $s \geq 0$, $u^i \in U$ and $n_i > 0$, it follows from induction that $u_{-n-2}V(L) \subset B$ for every $u \in V(L)$ and $n \in \mathbb{N}$. Then from the definition of C_2 , we have $C_2(V(L)) \subset B$. Hence $C_2(V(L)) = B$.

Therefore, $P_2(V(L)) = V(L)/B$ is spanned by

$$u^1(-1) \cdots u^k(-1)\mathbf{1} + C_2(V(L)) \quad \text{for } s \geq 0, u^i \in U.$$

Set

$$(5.14) \quad \mathfrak{g} = \{u + C_2(V(L)) \mid u \in U\}.$$

Then \mathfrak{g} is a Lie subalgebra of $P_2(V(L))$ because from the bracket formula (3.12), for $u, v \in U$,

$$\begin{aligned}
[u + C_2(V(L)), v + C_2(V(L))] &= u_0v + C_2(V(L)) \\
&= (u_0v)_{-1}\mathbf{1} + C_2(V(L)) \\
&= [u(0), v(-1)]\mathbf{1} + C_2(V(L)) \\
&= \sum_{i, l_i=0} \binom{-1-l_i}{k_i} (-1)^{k_i} f_i(u, v) (-1-l_i-k_i)\mathbf{1} + C_2(V(L)) \\
&= \sum_{i, l_i=k_i=0} f_i(u, v) (-1)\mathbf{1} + C_2(V(L)).
\end{aligned}$$

Recall Proposition 3.4 (3). Then we may get a basis of L^- from $u^0(-1), u'(-n)$ for $u^0 \in (U^0)'$, $u' \in U'$, $n \geq 1$ by choosing a basis of $(U^0)'$ and a basis of U' . Then from PBW theorem and $B = C_2(V(L))$ we immediately have:

LEMMA 5.10. *Let $U' \subset U$ be such that $U = (U^0 + \text{im } d) \oplus U'$. Then there is a subspace $(U^0)'$ of U^0 such that the map*

$$\begin{aligned}
&(U^0)' \oplus U' \rightarrow \mathfrak{g} \subset P_2(V(L)) \\
(5.15) \quad &u \mapsto u + C_2(V(L))
\end{aligned}$$

is a linear isomorphism. Furthermore, if $\ker d \cap \text{im } d = 0$, we may take $(U^0)' = U^0$.

Now we are in the position to prove the following:

PROPOSITION 5.11. *Let L be a local vertex Lie algebra over U , let $V(L)$ be the vertex algebra associated with L and let \mathfrak{g} be the Lie algebra defined by (5.14). Then $P_2(V(L))$ is isomorphic to $S(\mathfrak{g})$ as a Poisson algebra.*

PROOF. The identity map on \mathfrak{g} induces an onto homomorphism of Poisson algebras from $S(\mathfrak{g})$ to $P_2(V(L))$ by the universal mapping property of $S(\mathfrak{g})$. Since $V(L)$ is isomorphic to $U(L^-)$ as a vector space, it follows immediately from PBW theorem and $B = C_2(V(L))$ that $S(\mathfrak{g})$ is, in fact, isomorphic to $P_2(V(L))$. \square

We continue our discussion on the Lie algebra \mathfrak{g} . Recall from Section 3 that $L^0 = \{u(0) \mid u \in U\} \subset L$. It follows from Proposition 3.4 (3) and Lemma 5.10 that there is a linear map σ from \mathfrak{g} onto L^0 by sending $u + C_2(V(L))$ to $u(0)$ for $u \in U$. Since

$$[u(0), v(0)] = \sum_{i, l_i=k_i=0} f_i(u, v)(0) + C_2(V(L)),$$

σ in fact is a Lie algebra homomorphism. In general, σ is not an isomorphism. For example, if $u \in U^0$ and $u \notin \text{im } d$ then $u \notin C_2(V(L))$ and $u(0) = 0$.

We need the following lemma:

LEMMA 5.12. *Let V be a vertex (operator) algebra and let I be an ideal of V . Then $\bar{I} = (I + C_2(V))/C_2(V)$ is a Poisson ideal of $P_2(V)$ and $P_2(V/I) = P_2(V)/\bar{I}$.*

PROOF. Clearly, \bar{I} is a Poisson ideal of $P_2(V)$. Furthermore, by definition we have

$$P_2(V/I) = (V/I)/C_2(V/I), \quad C_2(V/I) = (C_2(V) + I)/I.$$

Then $P_2(V/I) = P(V)/\bar{I}$. \square

COROLLARY 5.13. *Let λ be a linear character of U^0 . Then the Poisson algebra $P_2(V(L, \lambda))$ is isomorphic to $S(\mathfrak{g})/J$ where J is the ideal generated by $(u - \lambda(u)) + C_2(V(L))$ for $u \in U^0$.*

PROOF. $V(L, \lambda)$ is a quotient vertex algebra of $V(L)$ modulo the ideal I generated by $u(-1) - \lambda(u)$ for $u \in U^0$. Then $P_2(V(L, \lambda))$ is isomorphic to $S(\mathfrak{g})/\bar{I}$ by Lemma 5.12. Clearly, $J = \bar{I}$ is generated by $u - \lambda(u) + C_2(V(L))$ for $u \in U^0$. \square

Recall that for a vertex algebra V , $L = g(V)$ is a vertex Lie algebra over V and that $g(V)^0 = V/DV$.

COROLLARY 5.14. *Let V be a vertex algebra and let ℓ be a complex number. Then $P_2(V^{[\ell]})$ is isomorphic to the Poisson algebra $S(V/DV)/J$ where J is the ideal of $S(V/DV)$ generated by $\mathbf{1} - \ell + DV$.*

PROOF. First we deal with the case $\ell = 0$. Then \mathfrak{g} in this case is isomorphic to V/DV and $P_2(V^{[0]})$ is isomorphic to $S(V/DV)$ by Proposition 5.11. For general ℓ , the result follows from Corollary 5.13 and the fact that $V^{[\ell]}$ is a quotient vertex algebra of $V^{[0]}$ modulo the ideal generated by $\mathbf{1}(-1) - \ell$. \square

Let \mathfrak{g} be any Lie algebra. Set $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Then $L(\mathfrak{g})$ is a local vertex Lie algebra with bracket formula

$$(5.16) \quad [a(m), b(n)] = [a, b](m+n) \quad \text{for } a, b \in \mathfrak{g}.$$

Then $V(L(\mathfrak{g})) = U(\hat{\mathfrak{g}}_-)$ is a vertex algebra. Note that in this case the \mathfrak{g} in Proposition 5.11 is isomorphic to the Lie algebra \mathfrak{g} we begin with. It follows from Proposition 5.11 that $P_2(V(L(\mathfrak{g})))$ is isomorphic to $S(\mathfrak{g})$ as a Poisson algebra. Then we have:

COROLLARY 5.15. *Let \mathfrak{g} be any Lie algebra. Then there exists a vertex algebra V such that $P_2(V)$ is isomorphic to $S(\mathfrak{g})$ as a Poisson algebra.*

Our next goal is to compute $P_2(M(\ell, 0))$ and $P_2(L(\ell, 0))$ for affine vertex operator algebras $M(\ell, 0)$ and $L(\ell, 0)$ defined below. Let \mathfrak{g} be a simple Lie algebra and H be a fixed Cartan subalgebra. Denote by θ the longest root of \mathfrak{g} . Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the affine Lie algebra. Set $\hat{\mathfrak{g}}_{\pm} = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}]$. Then $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_-$. For a complex number ℓ , set

$$(5.17) \quad M(\ell, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ + \mathfrak{g} + \mathbb{C}c)} \mathbb{C},$$

the generalized Verma $\hat{\mathfrak{g}}$ -module of level ℓ . Let $L(\ell, 0)$ be the irreducible quotient module of $M(\ell, 0)$. Then $M(\ell, 0)$ is a vertex operator algebra if ℓ is not negative the dual Coxeter number (cf. [FZ], [Li2]) and $L(\ell, 0)$ is a simple vertex operator algebra (cf. [DL], [FZ], [Li2]).

PROPOSITION 5.16. *The Poisson algebra $P_2(M(\ell, 0))$ is isomorphic to $S(\mathfrak{g})$. If ℓ is a positive integer, then $P_2(L(\ell, 0))$ is isomorphic to the quotient Poisson algebra of $S(\mathfrak{g})$ modulo the Poisson ideal generated by $e_{\theta}^{\ell+1}$.*

PROOF. From the proofs of Proposition 5.11 and Corollary 5.15 we see that $P_2(M(\ell, 0))$ is isomorphic to $S(\mathfrak{g})$.

From the structure of $L(\ell, 0)$ (cf. [K1]) we have $L(\ell, 0) = M(\ell, 0)/I$, where $I = U(\hat{\mathfrak{g}}_-)U(\mathfrak{g})e_{\theta}^{\ell+1}(-1)\mathbf{1}$ and $\mathbf{1}$ is a fixed highest weight vector. Note that I is an ideal of the vertex operator algebra $M(\ell, 0)$. Then $P_2(L(\ell, 0)) = P_2(M(\ell, 0))/\bar{I}$ by

Lemma 5.12 and $\bar{I} = (I + C_2(M(\ell, 0))) / C_2(M(\ell, 0))$ is a Poisson ideal of $S(\mathfrak{g})$. Since the associative product is defined as $a_{-1}b = a(-1)b$, the Lie product is defined as $a_0b = a(0)b$ and $a(-n-2)b \in C_2(L(\ell, 0))$ for $a \in \mathfrak{g}$, $n \in \mathbb{N}$, $b \in L(\ell, 0)$, it is clear that \bar{I} is a subset of the Poisson ideal generated by $e_\theta^{\ell+1}$ of $S(\mathfrak{g})$. Thus \bar{I} is the Poisson ideal generated by $e_\theta^{\ell+1}$. This concludes the proof. \square

Finally we compute $P_2(V_L)$ for lattice vertex operator algebra V_L . Let L be a positive definite even lattice of rank ℓ . Set

$$(5.18) \quad C_2(L) = \{\alpha \in L \mid \langle \alpha - \beta, \beta \rangle \leq 0 \text{ for all } \beta \in L\}.$$

Then $|\langle \alpha, \beta \rangle| \leq \langle \beta, \beta \rangle$ for every $\beta \in L$ if $\alpha \in C_2(L)$. Let $\alpha_1, \dots, \alpha_n$ be a \mathbb{Z} -basis for L and let $\lambda_1, \dots, \lambda_\ell$ be the dual basis for the dual lattice L° . Let k be a positive integer such that $kL^\circ \subset L$. Then for $\alpha \in C_2(L)$, we have $|\langle \alpha, \lambda_i \rangle| \leq k\langle \lambda_i, \lambda_i \rangle$ for $i = 1, \dots, \ell$. This implies that $C_2(L)$ is finite. It follows from the definition that $0 \in C_2(L)$ and $-\alpha \in C_2(L)$ if $\alpha \in C_2(L)$. It also follows from the definition that $k\alpha \notin C_2(L)$ for $0 \neq \alpha \in L$, $k \neq 0, \pm 1$. Furthermore, $C_2(L)$ spans L over \mathbb{Z} . If not, let α be an element of $L - \mathbb{Z}C_2(L)$ with the smallest length. Since $\alpha \notin C_2(L)$, there is a $\beta \in L$ such that $\langle \alpha - \beta, \beta \rangle \geq 1$. This implies that $\alpha - \beta \neq 0$, $\beta \neq 0$. From

$$(5.19) \quad \langle \alpha, \alpha \rangle = \langle \alpha - \beta, \alpha - \beta \rangle + 2\langle \alpha - \beta, \beta \rangle + \langle \beta, \beta \rangle,$$

we get $|\alpha| > |\beta|, |\alpha - \beta|$. From the choice of α , we must have $\beta, \alpha - \beta \in \mathbb{Z}C_2(L)$. Thus $\alpha \in \mathbb{Z}C_2(L)$. This is a contradiction.

REMARK 5.17. In [KL], a subset $\Phi(L)$ of L similar to $C_2(L)$ was introduced and it was proved to satisfy the same properties.

If $\alpha \in L - C_2(L)$, from the above argument and (5.19) we have $\langle \alpha, \alpha \rangle \geq 6$. Therefore $L_2 \cup L_4 \subset C_2(L)$, where $L_n = \{\alpha \in L \mid \langle \alpha, \alpha \rangle = n\}$. One can easily see that for $L = \mathbb{Z}\alpha$, we have $C_2(L) = \{0, \pm\alpha\}$.

We shall need the explicit construction of V_L given in [FLM] including the group \hat{L} and notation ι . Let e be a section from L to \hat{L} such that $e(0) = 1$ and $\epsilon(\cdot, \cdot)$ the corresponding 2-cocycle. Set

$$(5.20) \quad \mathbf{h} = \mathbb{C} \otimes_{\mathbb{Z}} L, \quad \hat{\mathbf{h}} = \mathbf{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}.$$

As a vector space,

$$(5.21) \quad V_L = S(\hat{\mathbf{h}}^-) \otimes \mathbb{C}_\epsilon[L],$$

where $\hat{\mathbf{h}}^- = \mathbf{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\mathbb{C}_\epsilon[L]$ is the ϵ -twisted group algebra of L .

For short we simply write e^α for $\iota(\alpha)$ for $\alpha \in L$. Then $e^\alpha e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta}$ for $\alpha, \beta \in L$. For $\alpha, \beta \in L$, from [FLM] we have

$$(5.22) \quad Y(e^\alpha, z)e^\beta = \epsilon(\alpha, \beta)z^{\langle \alpha, \beta \rangle} \exp\left(\sum_{m=1}^{\infty} \frac{\alpha(-m)}{m} z^m\right) e^{\alpha+\beta}.$$

Then

$$(5.23) \quad (e^\alpha)_j e^\beta = 0 \quad \text{for } j > -1 - \langle \alpha, \beta \rangle$$

$$(5.24) \quad (e^\alpha)_{-\langle \alpha, \beta \rangle - 1} e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta}.$$

Assume $\alpha \notin C_2(L)$. Then $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in L$ with $\langle \beta_1, \beta_2 \rangle \geq 1$. From (5.24) we have

$$\epsilon(\beta_1, \beta_2) e^\alpha = (e^{\beta_1})_{-1-\langle \beta_1, \beta_2 \rangle} e^{\beta_2} \in C_2(V_L),$$

noting that $-1 - \langle \beta_1, \beta_2 \rangle \leq -2$. Therefore

$$(5.25) \quad e^\alpha \in C_2(V_L) \quad \text{for } \alpha \in L - C_2(L).$$

Since $\alpha(-m)V_L \subset C_2(V_L)$ for $m \geq 2$, from (5.22) we have

$$(5.26) \quad Y(e^\alpha, z)e^\beta \equiv \epsilon(\alpha, \beta)z^{\langle \alpha, \beta \rangle}e^{\alpha(-1)z}e^{\alpha+\beta} \pmod{C_2(V_L)}.$$

For $\alpha \in L$, set

$$(5.27) \quad X_\alpha = e^\alpha + C_2(V_L), \quad Z_\alpha = \alpha(-1) + C_2(V_L) \in P_2(V_L).$$

Then

$$(5.28) \quad Z_{\alpha+\beta} = Z_\alpha + Z_\beta \quad \text{for } \alpha, \beta \in L.$$

LEMMA 5.18. *The following relations hold in $P_2(V_L)$ for $\alpha, \beta \in C_2(L)$:*

$$(5.29) \quad X_\alpha X_\beta = 0 \quad \text{if } \alpha + \beta \notin C_2(L);$$

$$(5.30) \quad X_\alpha X_\beta = \frac{1}{(-\langle \alpha, \beta \rangle)!} \epsilon(\alpha, \beta) Z_\alpha^{-\langle \alpha, \beta \rangle} X_{\alpha+\beta} \quad \text{if } \alpha + \beta \in C_2(L);$$

$$(5.31) \quad Z_\alpha^{1-\langle \beta-\alpha, \alpha \rangle} X_\beta = 0.$$

In particular,

$$(5.32) \quad Z_\alpha^{1+\langle \alpha, \alpha \rangle} = 0;$$

$$(5.33) \quad Z_\alpha X_\alpha = 0.$$

PROOF. If $\alpha + \beta \notin C_2(L)$, we have $e^{\alpha+\beta} \in C_2(V_L)$. Since $\alpha(-m)C_2(V_L) \subset C_2(V_L)$ for $m \geq 1$, from (5.26) we have

$$(5.34) \quad Y(e^\alpha, z)e^\beta \equiv 0 \pmod{C_2(V_L)}.$$

Then the relation (5.29) follows immediately.

If $\alpha + \beta \in C_2(L)$, we have

$$\langle \alpha, \beta \rangle = \langle (\alpha + \beta) - \beta, \beta \rangle \leq 0.$$

Using (5.26) we get

$$(5.35) \quad (e^\alpha)_{-1}e^\beta \equiv \frac{1}{(-\langle \alpha, \beta \rangle)!} \epsilon(\alpha, \beta) \alpha(-1)^{-\langle \alpha, \beta \rangle} e^{\alpha+\beta} \pmod{C_2(V_L)}.$$

This proves (5.30).

Using (5.26) again we get

$$(e^\alpha)_{-2}e^{\beta-\alpha} \equiv \frac{1}{(1-\langle \alpha, \beta-\alpha \rangle)!} \epsilon(\alpha, \beta-\alpha) \alpha(-1)^{1-\langle \alpha, \beta-\alpha \rangle} e^\beta \pmod{C_2(V_L)}.$$

Since $e_{-2}^\alpha e^{\beta-\alpha} \in C_2(V_L)$, from this we immediately obtain (5.31). Clearly, (5.32) and (5.33) are special cases of (5.31) with $\beta = 0, \alpha$. \square

Let $P(L)$ be the commutative associative algebra with identity generated by symbols $X_\alpha, Z_\alpha, \alpha \in C_2(L)$ with defining relations (5.28)-(5.33). (With $C_2(L)$ being finite, it is clear that $P(L)$ is finite-dimensional.)

PROPOSITION 5.19. *The commutative associative algebra $P_2(V_L)$ is isomorphic to the commutative associative algebra $P(L)$.*

PROOF. From the proof of Lemma 5.18 we see that $C_2(V_L)$ contains the following subspaces:

$$(5.36) \quad \alpha(-m-2)V_L \quad \text{for } \alpha \in L, m \in \mathbb{N};$$

$$(5.37) \quad S(\hat{\mathbf{h}}^-) \otimes e^\beta \quad \text{for } \beta \notin C_2(L);$$

$$(5.38) \quad S(\hat{\mathbf{h}}^-)\alpha(-1)^{1-\langle\beta-\alpha,\alpha\rangle} \otimes e^\beta$$

for $\alpha \in L, \beta \in C_2(L)$. (Notice that $\langle\beta-\alpha,\alpha\rangle \leq 0$ for $\beta \in C_2(L)$.) Let U be the space of V_L spanned by the above subspaces. Noticing that $C_2(L)$ spans L over \mathbb{Z} , we see that $P_2(V_L)$ as an associative algebra is generated by X_α, Z_α for $\alpha \in C_2(L)$ with the relations (5.28)-(5.33).

To show that there is no more relation, we prove that $U = C_2(V_L)$. First, notice that the following holds: $\alpha(-k-1)U \subset U$ for $\alpha \in L, k \in \mathbb{N}$. Because of (5.36) (5.26) is true with mod $C_2(V_L)$ being replaced by mod U . Then because of (5.38) it is clear that $(e^\alpha)_{-n-2}e^\beta \in U$ for any $\alpha, \beta \in L, n \in \mathbb{N}$.

Second, since $[\alpha(-k), (e^\beta)_{-n-2}] = \langle\alpha, \beta\rangle(e^\beta)_{-k-n-2}$ for $\alpha, \beta \in L, k, n \in \mathbb{N}$, using induction we can prove

$$(e^\alpha)_{-n-2}\alpha_{i_1}(-k_1) \cdots \alpha_{i_r}(-k_r)e^\beta \in U$$

for any $\alpha, \alpha_i, \beta \in L, k_1, \dots, k_r \geq 1$.

Third, similar to the proof of Proposition 5.11 using induction again we can prove that

$$(\alpha_{j_1}(-s_1) \cdots \alpha_{j_t}(-s_t)e^\alpha)_{-n-2}\alpha_{i_1}(-k_1) \cdots \alpha_{i_r}(-k_r)e^\beta \in U$$

for any $\alpha, \beta \in L, k_1, \dots, k_r \geq 1$. Thus $u_{-n-2}v \in U$ for $u, v \in V_L, n \geq 0$, so that $C_2(V_L) \subset U$. Therefore $U = C_2(V_L)$.

From this, there exists a subset

$$G \subset \langle\alpha(-1) \mid \alpha \in C_2(L)\rangle \otimes \{e^\alpha \mid \alpha \in C_2(L)\} \subset V_L$$

such that $G \cap C_2(V_L) = \emptyset$ and $V_L = C_2(V_L) \oplus \mathbb{C}G$, where for a set S , $\langle S \rangle$ denotes the free abelian semigroup (with identity) generated by S . Then G gives rise to a basis of $P_2(V_L)$.

In view of Lemma 5.18, we have an algebra homomorphism ψ from $P(L)$ onto $P_2(V_L)$ such that

$$(5.39) \quad \psi(Z_\alpha) = \alpha(-1) + C_2(V_L), \quad \psi(X_\alpha) = e^\alpha + C_2(V_L) \quad \text{for } \alpha \in C_2(L).$$

Let \tilde{G} be the corresponding subset of $P(L)$ with $\alpha(-1)$ and e^α being replaced by Z_α and X_α , respectively. Since $\psi(\tilde{G}) = G$ in $P_2(V_L)$ and G gives rise to a basis of $P_2(V_L)$, \tilde{G} is linearly independent. It follows from the relations (5.28)-(5.33) that \tilde{G} linearly spans $P(L)$. Then it follows immediately that ψ is a linear isomorphism. Therefore, $P_2(V_L)$ is isomorphic to $P(L)$ as an algebra. \square

COROLLARY 5.20. $P_2(L)$ has a Poisson algebra structure such that for $\alpha, \beta \in C_2(L)$,

$$(5.40) \quad \{Z_\alpha, Z_\beta\} = 0, \quad \{Z_\alpha, X_\beta\} = \langle\alpha, \beta\rangle X_\beta,$$

$$(5.41) \quad \{X_\alpha, X_\beta\} = 0 \quad \text{if } \langle\alpha, \beta\rangle \geq 0;$$

$$(5.42) \quad \{X_\alpha, X_\beta\} = \frac{1}{(-\langle\alpha, \beta\rangle - 1)!} Z_\alpha^{-\langle\alpha, \beta\rangle - 1} \epsilon(\alpha, \beta) X_{\alpha+\beta}$$

if $\alpha + \beta \in C_2(L)$ and $\langle\alpha, \beta\rangle \leq -1$.

PROOF. It follows from a calculation for the Lie brackets in $P_2(V_L)$. \square

REMARK 5.21. Let L be a nondegenerate even lattice that is not positive-definite. That is, there is an $\alpha \in L$ such that $\langle \alpha, \alpha \rangle = k < 0$. Then $(e^\alpha)_{k-1} e^{-\alpha} \in 1 + C_2(V_L)$. Thus $1 \in C_2(V_L)$. Therefore, $P_2(V_L) = 0$.

Let $L = \mathbb{Z}\alpha$ be a one-dimensional lattice such that $|\alpha| = 2k$, where k is a fixed positive integer. Then $C_2(L) = \{\alpha, 0, -\alpha\}$. Let B_k be the quotient algebra of the polynomial algebra $\mathbb{C}[X, Y, Z]$ modulo the relations $X^2 = Y^2 = XZ = YZ = 0$, $XY = \frac{1}{(2k)!} Z^{2k}$. Then A_k is the algebra over the curve $X^2 = Y^2 = XZ = YZ = 0$, $XY = \frac{1}{(2k)!} Z^{2k}$. Define $\{Z, X\} = 2kX$, $\{Z, Y\} = -2kY$, $\{X, Y\} = \frac{1}{(2k-1)!} Z^{2k-1}$. As an immediate consequence we have:

COROLLARY 5.22. *The above defined B_k is a Poisson algebra and it is isomorphic to $P_2(V_L)$.*

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